On Hadamard Embeddability

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Dedicated to Alston S. Householder
on the occasion of his seventy-fifth birthday.

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ABSTRACT

Given any k vectors of dimension $n>k$ which are mutually orthogonal, it is well known that this matrix can be completed to an $n\times n$ orthogonal matrix. Hadamard matrices form a subclass of orthogonal matrices. By contrast it is shown that it is possible to construct Hadamard submatrices with $2t+2$ rows that cannot be completed to a Hadamard matrix of order $4t$ for infinitely many values of t. Some familiarity with Hasse-Minkowski invariants is assumed. A large number of unsolved problems in this area are pointed out.

1. HADAMARD MATRICES

An Hadamard matrix H is a square matrix of order $4t$ such that the first row and column are all ones, otherwise $h_{ij}=1$ or -1, and $HH^T=4tI$, I the identity. (This is sometimes referred to as a normalized Hadamard matrix, but it is the only type we shall consider).

The implication of the orthogonal character of H (Hadamard) matrices is that any row other than the first has row sum zero, and between any two rows the ordered pairs of corresponding entries (i.e. in the same column) $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$ each occur t times. A similar statement is also true for columns. The reason for the designation is that H matrices satisfy the Hadamard determinant theorem with strict equality.

We say H is skew Hadamard if it admits a solution of the form $I+S$ where S is skew. To normalize, all rows other than the first must have their signs altered. Skew solutions are known for all cases where the order is <116.

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It has been known since the time of Paley (1933) [8] that (i) a skew solution can be found if $p^r+1\equiv 0$ (4), p a prime, and (ii) a solution exists for $2(p^r+1)$ if p is prime and $p^r+1\equiv 2$ (4). There are a number of other more complex criteria, but these were sufficient to establish the existence of an (in some sense) surprisingly large set of H matrices. In fact the only H matrices of the form $16t\pm 4$ that are not ruled out by these two results and are of order not exceeding 500 are 92, 116, 156, 172, 188, 236, 260, 268, 292, 356, 404, 428, 436.

It has not been easy to settle these cases. For an expanded discussion with the history of events leading to closing the gaps so that now 268 is the smallest unknown case, see [7, p. 1187]. Here we are content merely to mention that it was not until 1944 that 172 was settled, and nearly 20 years passed before the next case, 92, was disposed of.

Another quite difficult problem is to decide whether two H matrices of the same order are inequivalent. We say P and Q are signed permutation matrices if each row and column contains one entry either 1 or -1, with all remaining entries 0. Then if P and Q can be found such that $PHQ=G$, we say H and G are in the same equivalence class. Hedayat and Wallis discuss this problem on p. 1188 of [7]. We shall not discuss this problem except peripherally. Some of our methods undoubtedly give new constructions to some of the known cases.

2. DESIGNS

We say N is the incidence matrix of a balanced incomplete block design (BIB) with parameters v, r, k, b, λ if (i) $n_{ij}=0$ or 1, (ii) N is $v\times b$, (iii) the row sum is r and the column sum is k, (iv) $NN^T=(r-\lambda)I+\lambda J$, where J is a matrix of all ones. Evidently necessary conditions for such an N to exist are that $bk=rv$ and $(v-1)\lambda=r(k-1)$. If $b=v$, we say the BIB is symmetric, and we write the parameters as v, k, λ in that order. Any N associated with a symmetric design has the normal property $NN^T=N^TN$.

By virtue of the normal property, it follows that if a single column (block) is removed from N and all rows (varieties) of this column containing a one are also deleted, we obtain a new BIB, called the derived design, with parameters $v-k$, k, $k-\lambda$, $v-1$, λ.

In particular, if the symmetric design $4t-1$, $2t-1$, $t-1$ exists, then there is an H matrix H of order $4t$, and conversely. For if H exists, we may delete the first row and column and replace -1 by 0. Clearly the number of $(1,1)$ pairs between any two rows is now $t-1$. Conversely, in the incidence matrix for the design, replace each 0 by -1 and adjoin a row and column of ones to secure H of order $4t$. Accordingly we call such designs Hadamard designs (H designs). The parameters of the H derived design are $2t$, $2t-1$, t, $4t-2$, $t-1$.

Generally speaking, there are solutions of the derived design which are not embeddable in the parent design. Information here is scanty. For $\lambda=1$, the case of the finite projective plane, the derived design is merely the Euclidean plane, and it is always embeddable. Connor [2] proved a similar result for $\lambda=2$ in a remarkable paper. But for $\lambda=3$, Bhattacharya [1] found a solution for the derived design with parent $v=b=25$, $r=k=9$ that was nonembeddable. A convenient reference for these results is [5, pp. 252–264]. However, no case is known where the derived design exists but the parent fails to exist.

There are in fact two derived designs. If we partition the incidence matrix of the parent in the form

$$\begin{pmatrix} E & N_1 \\ 0 & N_2 \end{pmatrix},$$

where E is a column vector with each component 1 whereas 0 is a column vector with each component 0, then N_1 and N_2 are designs (with E and 0 of dimension r and $v-r$) with parameters

$$\begin{array}{ccccc} r & r-1 & \lambda & v-1 & \lambda-1 \\ v-r & r & k-\lambda & v-1 & \lambda \end{array}.$$

For small r and λ, clearly the first design might exist, but both the second and the parent could fail to exist. This in fact does occur: e.g. $v=22$, $r=7$, $\lambda=2$, which does not exist.

But with larger r and λ, more structure is demanded, so that for the Hadamard case with $r=2t-1$ and $v-r=2t$ the limitations are about equal. Further it is well known that there is a solution to the first design easily constructed from the second: arrange the solution to the second design in partitioned form as

$$\begin{array}{cc} 1 \quad 1 \quad \cdots \quad 1 & 0 \quad 0 \quad \cdots \quad 0 \\ N_1 & N_2 \end{array}.$$

Then N_1 together with N_2^* forms a solution to the first design where $N_2+N_2^*=J$, the matrix of all ones.

With this close relationship between the two derived designs, it would appear that a reasonable conjecture might be that, in the Hadamard case, the existence of either derived design would imply the existence of the symmetric parent. This is evidently true for the case $t=3$, since the derived designs form but a single equivalence class.

The case $t=5$ has been carefully studied by Singhi [10]. He found 6 inequivalent solutions for the parent design but 21 inequivalent solutions to the derived design with parameters.

$$10 \quad 9 \quad 5 \quad 18; \quad 4.$$

However, each of these solutions can be completed to the parent design, i.e., they are embeddable. The number of embeddable equivalence classes is surprisingly large.

The best result on the embedding problem is due to Verheiden [11], who proved that if all but 7 rows of a Hadamard matrix are constructible, then this configuration is Hadamard embeddable. In the concluding paragraph, we show that the Verheiden bound is best possible.

By a skew H design we mean that there is an incidence structure N such that $N+N^T=J-I$. We also require group-divisible partially balanced incomplete block designs, which we will define via the incidence matrix. Here $v=mn$, and we demand that the $v\times b$ incidence matrix have these properties: (i) row sum is a constant r and column sum is a constant k, and (ii) NN^T can be partitioned into m^2 $n\times n$ submatrices such that the "diagonal" submatrix is $(r-\lambda_1)I+\lambda_1 J$ while the "off diagonal" submatrices have the form $\lambda_2 J$. The groups of rows (varieties) $1,\ldots,n$; $n+1,\ldots,2n$; etc. are called first associates. Two varieties which are not first associates are called second associates and are of course characterized by the fact that their row inner product is λ. For more complete detail, the reader is referred to [3, pp. 7–14].

There are two necessary conditions that must always be satisfied here: (i) $rv=kb$ and (ii) $r(k-1)=(n-1)\lambda_1+n(m-1)\lambda_2$, and we shall always write the parameters in the order

$$v \quad r \quad k \quad b; \; m \quad n \quad \lambda_1 \quad \lambda_2.$$

If we can express each of the parameters in terms of an auxiliary variable so that the above conditions are satisfied, we say we have a family of designs. We use the same terminology for BIB designs.

If N is the incidence matrix of our group divisible design, then dom $NN^T=rk$, and the other eigenvalues are $r-\lambda_1$ and $rk-v\lambda_2$. These designs are classified as singular (S), semiregular (SR), or regular (R) according as $r-\lambda_1=0$, $rk-v\lambda_2>0$; $r-\lambda_1>0$, $rk-v\lambda_2=0$; or $r-\lambda_1>0$, $rk-v\lambda_2>0$.

3. TWO FAMILIES

THEOREM 1. *If it is possible to construct* $2t+1$ *rows of an H matrix of order* $4t$ *(not using a row of all* $+$*), then the SR design*

$$4t+2 \quad 2t \quad 2t+1 \quad 4t; \quad 2t+1 \quad 2 \quad 0 \quad t$$

exists.

Proof. Write down the $2t+1$ rows, where each row contains $2t$ ones and $2t$ minus ones. Replace 1 by $\begin{pmatrix}1\\0\end{pmatrix}$ and -1 by $\begin{pmatrix}0\\1\end{pmatrix}$, regarding the rows so formed as a pair of first associates. Clearly $\lambda_1=0$, and, in some order, we have the combinations between two groups of first associates

$$\begin{matrix}1&1&0&0\\0&0&1&1\\1&0&1&0\\0&1&0&1\end{matrix}$$

with each arrangement occurring t times. Hence $\lambda_2=t$. ∎

It is well known that if $4t-1$ is a prime or a power of a prime, then the squares over such a field form a difference set mod $4t-1$. For 11, the squares are $1,3,4,5,9$. On forming the differences $3-1$, $4-1$, etc., every possible difference mod 11 occurs twice. Thus, if we form the initial row

$$0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0,$$

where zero is the index of the first element and one occurs precisely at the squares mod 11, i.e. $1,3,4,5,9$, we find that every difference occurs twice. Hence, upon cyclic advance, the inner product of any two rows is necessarily 2, yielding a cyclic solution to the Hadamard design with $v=11$, $r=5$. By contrast the next result seems virtually unknown.

THEOREM 2 (Eakin and Hasse). *For p an odd prime, the squares $0^2,1^2,\ldots,[(p-1)/2]^2$ mod $2p$ form a partial difference set* mod $2p$ *with each difference occurring $(p-1)/2$ times except for the difference p, which occurs $p-1$ times.*

Hasse [6, pp. 147–8] proves an analogue of this theorem, but it remained for Eakin [4] to note that here was in fact a partial difference set, although he made no applications of the result. To illustrate the use of the theorem, when $p=5$ the squares are $0,1,4,6,9$ mod 10, and we choose

$$1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1$$

as the initial row. Upon cyclic advance, rows $1,6$; $2,7$; …; $5,10$ have inner product 4, whereas any other pair of rows has inner product 2. In contrast with our definition, here $1,6$; $2,7$; etc. are first associates. But of course renaming the varieties can give our procedure.

The use of Theorem 2 enables us to assert that the symmetric R designs

$$4t+2 \quad 2t+1 \quad 2t+1 \quad 4t+2; \quad 2t+1 \quad 2 \quad 2t \quad t$$

exist if $2t+1$ is a prime. We shall prove a more general result later. However, these designs do not exist for all t. In fact for symmetric R designs, there is an application of Hasse-Minkowski invariant theory dealing with the rational congruence of symmetric rational matrices. This result states (see [9, p. 228]):

THEOREM 3. *For a symmetric R design to exist, it is necessary that*

$$\left(-1, r^2 - v\lambda_2\right)_p^{m(m-1)/2}\left(r^2 - v\lambda_2, vn^m\right)_p = 1,$$

where $(a,b)_p$ is the Hilbert normed residue symbol, provided $r-\lambda_1$ is a square.

We have purposely simplified the theorem for our case, since $r-\lambda_1=1$. Here $r^2 - v\lambda_2 = 2t+1$, and the expression reduces to $(-1, 2t+1)_p^{2t^2+t+1}$ If $t=2u$, we have the possibility $(-1, 4u+1)_p = -1$, and this in fact will occur unless $4u+1$ can be represented as a sum of two squares.

THEOREM 4. *The symmetric R design*

$$4t+2 \quad 2t+1 \quad 2t+1 \quad 4t+2; \quad 2t+1 \quad 2 \quad 2t \quad t$$

exists if $2t+1 = p^r$.

Proof. There are two cases. When $p^r \equiv -1$ (4), if we let a_i denote the elements of the field, and if we set $q_{ij} = \chi(a_i - a_j)$ where $\chi(a) = 1$ if a is a square and -1 if a nonresidue, we secure a skew symmetric matrix, where of course $\chi(0) = 0$, with the property that

$$QQ^T = p^r I - J,$$

a very familiar result which essentially goes back to Paley (1933) [8]. A most readable account of all of these details can be found in [5, p. 309].

Now let I, 0, J denote the 2×2 identity, null, and all one submatrices. Replace the 0 entry in the main diagonal of Q by I, the -1 entries by 0, and the $+1$ entries by J. Take 1,2; 3,4; etc. as first associates. Then clearly $\lambda_1 = 2t$, since there are t entries of $+1$ in each row of the original Q. Clearly $r = k = 2t+1$. Secondly, there are precisely $(p^r - 3)/4$ cases where we find the combination $(1,1)$ between corresponding entries of two rows of Q when $p^r \equiv -1$ (4). In our augmented version, we therefore have inner product $(p^r - 3)/2 + 1 = t$ for λ_2, because the skew property says that between two groups of first associates we have I paired against 0 and I paired against J, giving a contribution of one to the inner product.

In case $p^r \equiv 1$ (4), the Q matrix is symmetric, and there are two cases to discuss. If the diagonal 0 is paired against 1 between two rows, then elsewhere in these two rows the combination (1, 1) between corresponding elements occurs $(p^r-5)/4$ times. On expanding the Q matrix, we then have inner product of $(p^r-1)/2$. In the opposite case we have the pair (1, 1) occurring $(p^r-1)/4$ times with no contribution to the inner product from the diagonal terms. In either case the inner product is $\lambda_2 = t$. ■

We illustrate in full the solutions for $t=1$ and $t=2$ showing the cyclic advance:

$$\begin{bmatrix} I & J & 0 \\ 0 & I & J \\ J & 0 & I \end{bmatrix}, \qquad \begin{bmatrix} I & J & 0 & 0 & J \\ J & I & J & 0 & 0 \\ 0 & J & I & J & 0 \\ 0 & 0 & J & I & J \\ J & 0 & 0 & J & I \end{bmatrix}.$$

In fact what we have done establishes Theorem 2. We need merely write down the matrix

$$\begin{pmatrix} I+Q & Q \\ Q & I+Q \end{pmatrix}$$

by retaining in the first matrix all columns with odd subscript and transferring, in the same order, all even numbered columns to the second half.

4. SOME NEW HADAMARD DERIVED DESIGNS

Considering the two families of Sec. 3 with parameters

$$4t+2 \quad 2t \quad 2t+1 \quad 4t; \quad 2t+1 \quad 2 \quad 0 \quad t$$

$$4t+2 \quad 2t+1 \quad 2t+1 \quad 4t+2; \quad 2t+1 \quad 2 \quad 2t \quad t$$

we note that v, k, m, n are identical. If we therefore use the same association scheme in each, we may combine the two designs by writing their incidence matrices side by side to obtain the Hadamard derived design

$$4t+2 \quad 4t+1 \quad 2t+1 \quad 8t+2; \quad 2t$$

yielding:

THEOREM 5. *If* $2t+1=p^r$, *then there is a Hadamard derived design with the above parameters admitting a decomposition into an R and an SR subdivision.*

While the theorem can be generalized slightly, the above statement is sufficient for our purposes. It appears that if $t>2$, we obtain a number of inequivalent classes, since both the SR and the R admit inequivalent solutions for $t \geqslant 3$. We shall not explore this question, since our main purpose is to secure Hadamard derived designs that are not embeddable. Indeed, we conjecture that every solution to the derived design obtained in Sec. 3 is nonembeddable for odd $t>1$. This we have been unable to prove. In any event the conjecture is false for t even. In fact we have been able to construct embeddable solutions for t even, $t \leqslant 14$, $t \neq 10$, but these results are currently too fragmentary to discuss.

The case $t=3$ of our conjecture could be verified by a computer search, but there are in fact a large number of inequivalent solutions, so that it would be costly. For $t>3$ it would be an unrealistic method.

5. A VACUOUS RESULT

LEMMA. *The only manner in which our derived design can be completed to a Hadamard design is to adjoin* $4t$ *rows which have precisely* $2t-1$ *unities over the columns of the SR portion and one row which has* $4t$ *unities over these columns.*

Proof. Clearly we cannot have two unities in any adjoined row occurring within the same column subdivision of the R portion, for the column intersection is already $2t=\lambda$. Thus there are at least $2t-1$ unities in the columns of the SR section for any adjoined row.

Suppose some row has $2t+a$ unities in this portion, $a \geqslant 0$. Then there are $a+1$ column subdivisions which have a pair of zeros in the adjoined row in the two columns of the subdivision.

Consider a row subdivision of two rows. In the SR portion we automatically have a contribution of $2t+a$ to the inner product of the two rows, since we have always either the ordered pair $(0,1)$ or $(1,0)$ for the two rows. There will be at least one case where, within the column subdivisions of the R section, the 2×2 matrix I will occur opposite a pair $(0,0)$. Elsewhere the inner product contribution from R is necessarily even, forcing a to be even. If ever we have a pair $(0,1)$ or $(1,0)$ in a column subdivision of R in this adjoined row, then there is a case where this is paired against I, giving a contribution of exactly one to the inner product, which now forces a to be odd. Thus either $a=-1$ or $a=2t$. ∎

Letting E denote a vector of length $4t$ and 0 a vector of length $4t+2$, we therefore have schematically the completed extension of the form

$$\begin{array}{cc|c} 0^T & E^T & 1 \\ \hline \text{R design} & \text{SR design} & 0 \\ \hline \multicolumn{2}{c|}{\text{Design}} & E \end{array}$$

The initial row (excluding the last column) along with the $4t$ rows labelled Design form a derived Hadamard design.

The reason we are able to secure definitive theorems for the even case is that the SR construction can be based upon a duplicated solution to a Hadamard matrix. Thus, using 1 and 0 for the elements of the matrix,

$$H_{8t}=\begin{pmatrix} H_{4t} & H_{4t} \\ H_{4t} & H_{4t}^* \end{pmatrix}$$

where H_{4t}^* is the complement of Hadamard H_{4t}, i.e. $H_{4t}+H_{4t}^*=J$.

For the case $t=2$, we have:

$$H_8=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We choose the SR portion by rearranging the columns of the last group along with the second row to yield

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$$

Thus the columns are paired in such a way that each successive pair are complements (except for the initial row, where we have all ones followed by all zeros).

Clearly such an arrangement is always possible for any case where H_{4t} exists. Now assuming embeddability, consider the $8t$ columns that arise when the design is extended to the full Hadamard matrix in the SR division, where we are purposely excluding the adjoined row of all ones (the initial row of the schematic completion pattern). This submatrix is square, and we denote it by M. It is elementary to compute M^TM. There are two caveats. One needs to recall that in forming inner products the pairs $(0,0)$ and $(1,1)$ in the SR portion both yield a contribution of 1. Further there is a contribution of one from the omitted adjoined row. The completed Hadamard design has row sum $8t+1$ with $\lambda=4t$. Thus the diagonal term of M^TM is $4t-1$, and the off diagonal values are $4t-2$ between the "disjoint" pairs, $2t-2$ elsewhere among the first $4t$ columns as well as between two such columns in the last $4t$ columns, and $2t-1$ between columns of which one belongs to the first set of $4t$ columns and the other to the final $4t$ columns. Thus

$$M^TM=\begin{pmatrix} U & V \\ V & U \end{pmatrix}$$

where $V=(2t-1)J$ and

$$U=\begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & & \vdots \\ B & B & \cdots & A \end{bmatrix},$$

$$A=\begin{pmatrix} 4t-1 & 4t-2 \\ 4t-2 & 4t-1 \end{pmatrix}, \qquad B=(2t-2)J.$$

To compute the Hasse-Minkowski invariant, we need to find the eigenvalues of M^TM. One of these is the constant row sum, which is easily determined to be $(4t-1)^2=\operatorname{dom} M^TM$. The remaining eigenvalues are 1 with multiplicity $4t+1$, and $4t+1$ with multiplicity $4t-2$. (One need only observe that the eigenvalues of M^TM are the eigenvalues of $U+V$ and $U-V$. Both of these have the structural form of the matrix U, making it possible to secure the other two eigenvalues.) The result is of interest; technically we are dealing with a $D(3)$ design which normally has three distinct eigenvalues in addition to the dominant eigenvalue.

Our next step is to find a set of $4t+1$ linearly independent rational eigenvectors associated with $\lambda=1$. Evidently $E^T,\ -E^T$ is one such vector, where E is a row vector of length $4t$ of all ones. Secondly, let F_{2u} be a vector such that all components are 0 except for $f_{2u-1}=1$, $f_{2u}=-1$. Then it is readily verified that we have a set of eigenvectors meeting the above condition. Since the vectors are orthogonal, the Gramian Q is $t2^{4t+3}$ with square-free part $2t$. It is not necessary to find the Gramian associated with the eigenvalue $4t+1$ since its square-free part is clearly 1 inasmuch as we can take the vector E^T, E^T for the dominant eigenvector. These facts make the calculation of the Hasse-Minkowski invariant trivial. It is merely $(4t+1,-1)_p$. A complete discussion is found in [9, p. 226]. We therefore have the mildly interesting result that the technique of rational equivalence yields no information with respect to Hadamard embeddability, since the R design also fails to exist according to Theorem 3. Instead we secure a theorem about nonembeddability of SR designs and their duals in derived Hadamard designs. This result could possibly strengthen the conjecture that embeddability at this stage is always possible.

6. HADAMARD NONEMBEDDABILITY

Here we again consider the case of t even, but now we use the duplicated columns in

$$\begin{pmatrix} H_{4t} & H_{4t} \\ H_{4t} & H_{4t}^* \end{pmatrix}.$$

Omitting the row of ones, we write the duplicated columns in pairs choosing the first $2t$ pairs so that we have one in the initial row. We adjoin two rows of the form

$$\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix},$$

where the first choice is used for the first $2t$ pairs of columns and the second for the second $2t$ pairs of columns. Thus when $t=1$, we have

$$\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1
\end{array}$$

Supposing in general that this matrix is Hadamard embeddable and calling the $8t\times 8t$ matrix that must be adjoined to the $8t$ columns S, we have

$$S^TS=\begin{pmatrix} U & V \\ V & U \end{pmatrix},$$

where V is a constant matrix of order $4t$, $V = 2tJ$, and

$$U = \begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & & \vdots \\ B & B & \cdots & A \end{bmatrix},$$

$$A = \begin{pmatrix} 4t+1 & 4t-1 \\ 4t-1 & 4t+1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2t+1 & 2t-1 \\ 2t-1 & 2t+1 \end{pmatrix}.$$

Now if our matrix can be extended to the symmetric Hadamard design with row sum $8t+1$ and $\lambda = 4t$, the continuation of the above columns to the Hadamard design must yield column inner products of the above form with

$$\begin{pmatrix} \underline{U} & V \\ V & \underline{U} \end{pmatrix},$$

where correspondingly

$$\underline{A} = \begin{pmatrix} 4t & 1 \\ 1 & 4t \end{pmatrix}, \qquad \underline{B} = \begin{pmatrix} 2t-1 & 2t+1 \\ 2t+1 & 2t-1 \end{pmatrix}.$$

In the lemma of the preceding section, it was noted that in the extension, a single row of unities was adjoined. Hence the question remains whether

$$\begin{pmatrix} \underline{U} & V \\ V & \underline{U} \end{pmatrix} - J$$

is rationally equivalent to the identity matrix.

To compute the eigenvalues of the matrix above, on adding the lower subdivision to the upper rowwise and then subtracting the first from the second columnwise, we secure the matrices

$$\begin{bmatrix} R & S & \cdots & S \\ S & R & \cdots & S \\ \vdots & \vdots & & \vdots \\ S & S & \cdots & R \end{bmatrix}, \qquad \begin{bmatrix} P & Q & \cdots & Q \\ Q & P & \cdots & Q \\ \vdots & \vdots & & \vdots \\ Q & Q & \cdots & P \end{bmatrix},$$

$$R = \begin{pmatrix} 6t-2 & 2t-1 \\ 2t-1 & 6t-2 \end{pmatrix}, \qquad S = \begin{pmatrix} 4t-3 & 4t-1 \\ 4t-1 & 4t-3 \end{pmatrix},$$

$$P = \begin{pmatrix} 2t & -2t+1 \\ -2t+1 & 2t \end{pmatrix}, \qquad Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The eigenvalues are the dominant eigenvalue of the row sum, which is a square, and from the R, S matrix the values $4t+1$ and 1 of multiplicities

$2t-1$ and $2t$. For the P,Q matrix $4t+1$ and 1 have multiplicities $2t-1$ and $2t+1$. Again it is surprising that there are only three distinct eigenvalues.

It is easy to secure a set of orthogonal eigenvectors for the eigenvalue 1. We note, on subtracting J, that we have reduced $\underline{A}$, $\underline{B}$, and C to

$$\underline{A}-J=\begin{pmatrix}4t-1 & 0\\ 0 & 4t-1\end{pmatrix},\qquad \underline{B}-J=\begin{pmatrix}2t-2 & 2t\\ 2t & 2t-2\end{pmatrix},\qquad C-J=2t-1.$$

The row sums of these 2×2 submatrices [looking at C as a collection of 2×2 submatrices of the form $(2t-1)J$] are $4t-1$ for the "diagonal"submatrices and $4t-2$ for the "off diagonal" submatrices. We claim that if we choose any column of a Hadamard matrix of order $4t$ (except for the column of all ones) and replace each 1 by a pair $(1,1)$ and each -1 by $(-1,-1)$, then this is in fact an eigenvector associated with 1. Aside from the "diagonal" submatrix, the constancy of the sums must yield either $(4t-2)$ or $-(4t-2)$, since there are the same number of ones as their negatives in the presumed eigenvector. From the "diagonal" submatrix, we secure correspondingly $-(4t-1)$ or $4t-1$. Hence the linear transformation of such a vector produces a vector with only ± 1 as entries. Further, by the structure of the "diagonal" submatrix, these values are duplicated in pairs. Finally, since the determination of whether the linear transformation yields 1 or -1 depends solely on whether this "diagonal" term coincides with 1 or -1 under the transformation, this vector must go into itself under the transformation. There are $4t-1$ such eigenvectors.

The remaining two eigenvectors are secured by choosing the alternating series $1,-1,1,-1,\ldots$ for the first $4t$ terms and 0 thereafter, and the reverse with 0 for the first $4t$ terms followed by $1,-1\ldots$ for the next $4t$. For the odd numbered rows we get $4t-1$ from the "diagonal" and $-4t+2$ from the $2(t-1)$ groups of two in the upper half. But the signs are reversed for the even rows of the top half. The other vector behaves similarly with respect to the lower half. Hence the square-free part of the Gramian of these vectors is $2t$. The Gramian associated with the eigenvalues $4t+1$ must therefore be a square, since the scalar Gramian associated with the dominant eigenvalue is $8t$, and the product of the Gramians must be a perfect square.

The Hasse-Minkowski invariant becomes

$$(-1,4t+1)_p(4t+1,2t)_p.$$

Unlike the preceding case, this is quite restrictive. Thus embedding is impossible whenever $4t+1$ is a prime and t is odd. Note that

$$(4t+1,2t)_{4t+1}=(4t+1,4t)_{4t+1}(4t+1,2)_{4t+1}.$$

The first term is always 1 and the second always -1 for this case.

In fact embedding is ruled out for the cases

$$t=1,3,7,9,11,13,15,16,21,22.$$

By the time $t=5,8,14,17,19$ are excluded (since this form of the derived design itself does not exist then), this is an impressive collection of successes. Further, since Hasse-Minkowski theory would not appear to be an advisable technique, the evidence strongly suggests that embeddability is never possible from this particular construct. It would be highly desirable if this result could be established for all t.

We are therefore able to answer a question left open by Singhi [10], since the case $t=1$ yields a nonembeddable Hadamard derived design with parameters 10 9 5 18; 4. Moreover, Verheiden (unpublished) has been able to extend his bound in the Hadamard case to any symmetric BIB. Now we can always add one more row to our construction: merely adjoin a row of zeros to the columns of the R part, a row of ones to the columns of the SR part, and one column with 1 in this adjoined row, 0 elsewhere. With this technique, we have constructed 11 rows of the 19 rows of the symmetric BIB with parameters (19, 9, 4). Now if we could add one more row, then by Verheiden's result, only 7 rows would remain, making completion possible. Since this design is, however, nonembeddable, we have an array which is saturated in the sense that no further row can be added if the Hadamard character is to be preserved.

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